

On the Cohomology of Modular Lie Algebras

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Dedicated to Robert L. Wilson and James Lepowsky on the occasion of their 60th birthdays

Abstract

In this paper we establish a connection between the cohomology of a modular Lie algebra and its p -envelopes. We also compute the cohomology of Zassenhaus algebras and their minimal p -envelopes with coefficients in generalized baby Verma modules and in simple modules over fields of characteristic $p > 2$.

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Introduction

This paper is mainly a survey on the cohomology of modular Lie algebras but there are also several new results. It is intended to be sufficiently self-contained to serve as a first introduction to this topic which plays an important role in modular representation theory. Another goal is to advertise the use of p -envelopes and truncated (co)induced modules in the representation theory of Lie algebras over fields of prime characteristic p . As a main example throughout the paper we consider a series of rank one Lie algebras of Cartan type, the Zassenhaus algebras (which are simple in characteristic $p > 2$). In the following we will describe the contents of the paper in more detail.

In the first section we give some background material on p -envelopes which will be useful for the rest of the paper. We refer the reader to [27, 28, 29] for more details and most of the proofs. The second section is devoted to establishing a connection between the cohomology of a modular Lie algebra and its p -envelopes which is very similar to a factorization theorem of Hochschild and Serre in characteristic zero. In particular, we show that the vanishing of the cohomology of a modular Lie algebra is equivalent to the vanishing of the cohomology of any of its p -envelopes. Parts of this section are contained in an unpublished diploma thesis of Katrin Legler [24]. In the third section we reformulate a cohomological vanishing theorem of Dzhumadil'daev in terms of the universal p -envelope, a result that rests mainly on the trivial action of the universal p -envelope \hat{L} of L on the cohomology of L . We also derive several consequences which will be useful later in the paper. In particular, we generalize a cohomological vanishing theorem for \mathbb{Z} -graded Lie algebras due to Chiu

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and Shen. In the fourth section we introduce truncated (co)induced modules which are important in the classification of simple modules for certain classes of modular Lie algebras. We recall the important fact that truncated induced modules and truncated coinduced modules coincide up to a twist. In particular, duals of truncated (co)induced modules are again truncated (co)induced modules, respectively. Moreover, we give a short proof of Shapiro's lemma for truncated induced modules. Mil'ner used his version of truncated induced modules in order to classify the simple modules of the Zassenhaus algebras [25]. In this paper we restrict our attention to certain truncated induced modules of the Zassenhaus algebras, the so-called generalized baby Verma modules which are sufficient for computing the cohomology of Zassenhaus algebras with coefficients in simple modules. In the final section we reprove Dzhumadil'daev's cohomological reduction theorem for Zassenhaus algebras by using Shapiro's lemma for truncated induced modules. Then we apply this to compute the 1-cohomology of Zassenhaus algebras with coefficients in generalized baby Verma modules and in simple modules over fields of characteristic $p > 2$ without using Mil'ner's classification of the simple modules for the Zassenhaus algebras. As a consequence we determine the central extensions of the Zassenhaus algebras. We also apply the main result of Section 2 in order to compute the 1-cohomology of the minimal p -envelopes of the Zassenhaus algebras with coefficients in generalized baby Verma modules and in simple modules as well as their central extensions. The results for the minimal p -envelopes of the Zassenhaus algebras seem to be new.

For the notation and fundamental results from the representation theory of modular Lie algebras we refer the reader to [29] and [28].

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1 p -Envelopes

Let L be a Lie algebra over a field of prime characteristic p . A triple $(\mathfrak{L}, (\cdot)^{[p]}, \iota)$ is called a p -envelope of L if \mathfrak{L} is a restricted Lie algebra with p -mapping $(\cdot)^{[p]}$ and ι is a Lie algebra monomorphism from L into \mathfrak{L} such that the restricted subalgebra $\langle \iota(L) \rangle_p$ of \mathfrak{L} generated by $\iota(L)$ coincides with \mathfrak{L} .

Proposition 1.1 *Let L be a Lie algebra over a field of prime characteristic p and let $(\mathfrak{L}, (\cdot)^{[p]}, \iota)$ be a p -envelope of L . Then the following statements hold:*

- (1) $[\mathfrak{L}, \mathfrak{L}] \subseteq \iota(L)$.
- (2) *If \mathfrak{I} is a subspace of \mathfrak{L} such that $\iota(L) \subseteq \mathfrak{I}$, then \mathfrak{I} is an ideal of \mathfrak{L} .*

Proof. (1): Since $\langle \iota(L) \rangle_p = \mathfrak{L}$, it follows from [29, Proposition 2.1.3(2)] that

$$[\mathfrak{L}, \mathfrak{L}] = [\langle \iota(L) \rangle_p, \langle \iota(L) \rangle_p] = [\iota(L), \iota(L)] \subseteq \iota(L).$$

(2): By using (1), we obtain that $[\mathfrak{L}, \mathfrak{I}] \subseteq [\mathfrak{L}, \mathfrak{L}] \subseteq \iota(L) \subseteq \mathfrak{I}$. \square

Moreover, the following structural features are preserved by p -envelopes. In particular, p -envelopes of solvable (nilpotent, abelian) Lie algebras are solvable (nilpotent, abelian, respectively).

Proposition 1.2 *Let L be a Lie algebra over a field of prime characteristic p and let $(\mathfrak{L}, (\cdot)^{[p]}, \iota)$ be a p -envelope of L . Then the following statements hold:*

- (1) L is abelian if and only if \mathfrak{L} is abelian.
- (2) L is nilpotent if and only if \mathfrak{L} is nilpotent.
- (3) L is solvable if and only if \mathfrak{L} is solvable.

Proof. By definition $\langle \iota(L) \rangle_p = \mathfrak{L}$ and therefore the assertions are immediate consequences of [29, Proposition 2.1.3(2)]. \square

Remark. Note that every abelian Lie algebra is restrictable (e.g. by the trivial p -mapping) but neither nilpotent Lie algebras nor solvable Lie algebras are necessarily restrictable (cf. [2, Chapter I, §4, Exercise 24] as well as [29, Example 3 on p. 72 and Exercise 2.2.9]).

A p -envelope $(\mathfrak{L}_{\min}, (\cdot)^{[p]}, \iota)$ of a finite-dimensional modular Lie algebra L is called *minimal* if its dimension is minimal among the dimensions of all p -envelopes of L . Any two minimal p -envelopes are isomorphic as ordinary Lie algebras (cf. [29, Theorem 2.5.8(1)]). Of importance is the well-known fact that a p -envelope $(\mathfrak{L}, (\cdot)^{[p]}, \iota)$ of L is minimal if and only if the center $C(\mathfrak{L})$ of \mathfrak{L} is contained in $\iota(L)$ (cf. [29, Theorem 2.5.8(3)]).

It turns out that minimal p -envelopes of simple Lie algebras are simple as restricted Lie algebras (but in general not simple as ordinary Lie algebras since a non-restrictable Lie algebra is isomorphic to a non-zero proper ideal of any of its p -envelopes) and minimal p -envelopes of semisimple Lie algebras are semisimple.

Proposition 1.3 *Let L be a finite-dimensional Lie algebra over a field of prime characteristic p and let $(\mathfrak{L}_{\min}, (\cdot)^{[p]}, \iota)$ be a minimal p -envelope of L . Then the following statements hold:*

- (1) *If L is simple, then \mathfrak{L}_{\min} has no non-zero proper p -ideals.*
- (2) *If L is semisimple, then \mathfrak{L}_{\min} is semisimple.*

Proof. (1): Let \mathfrak{I} be a p -ideal of \mathfrak{L}_{\min} . Then $\mathfrak{I} \cap \iota(L)$ is an ideal of $\iota(L)$. But because $\iota(L)$ is simple, either $\mathfrak{I} \cap \iota(L) = 0$ or $\mathfrak{I} \cap \iota(L) = \iota(L)$.

In the first case it follows from Proposition 1.1(1) that $[\mathfrak{I}, \mathfrak{L}_{\min}] \subseteq \mathfrak{I} \cap [\mathfrak{L}_{\min}, \mathfrak{L}_{\min}] \subseteq \mathfrak{I} \cap \iota(L) = 0$, i.e., that $\mathfrak{I} \subseteq C(\mathfrak{L}_{\min}) \subseteq \iota(L)$ which implies that $\mathfrak{I} = \mathfrak{I} \cap \iota(L) = 0$.

In the second case one has $\iota(L) \subseteq \mathfrak{I}$. Since \mathfrak{I} is a p -ideal of \mathfrak{L}_{\min} , one obtains that $\mathfrak{L}_{\min} = \langle \iota(L) \rangle_p \subseteq \mathfrak{I}$, i.e., that $\mathfrak{I} = \mathfrak{L}_{\min}$.

(2): Let \mathfrak{I} be an abelian ideal of \mathfrak{L}_{\min} . Then $\mathfrak{I} \cap \iota(L)$ is an abelian ideal of $\iota(L)$. But because $\iota(L)$ is semisimple, $\mathfrak{I} \cap \iota(L) = 0$.

As in the proof of (1), one has $\mathfrak{I} \subseteq C(\mathfrak{L}_{\min}) \subseteq \iota(L)$. Since $\iota(L)$ is semisimple and \mathfrak{I} is an abelian ideal of $\iota(L)$, it readily follows that $\mathfrak{I} = 0$. \square

We conclude this section by introducing the Zassenhaus algebras and their minimal p -envelopes. Let \mathbb{F} be a field of prime characteristic p , let m be a positive integer, and let $A(m)$ denote the subalgebra of the algebra of divided powers over \mathbb{F} generated by $\{x^{(a)} \mid 0 \leq a \leq p^m - 1\}$ where $x^{(a)}x^{(b)} = \binom{a+b}{a}x^{(a+b)}$ for non-negative integers a and b . A derivation d of $A(m)$ is called *special* if $d(x^{(0)}) = 0$ and $d(x^{(a)}) = x^{(a-1)}d(x^{(1)})$ for every $1 \leq a \leq p^m - 1$. The Lie algebra $W(m) := \{d \in \text{Der}_{\mathbb{F}}(A(m)) \mid d \text{ is special}\}$ of special derivations of $A(m)$ for some positive integer m is called a *Zassenhaus algebra*. It is well-known that $W(m)$ is simple if and only if $p > 2$ (cf. [29, Theorem 4.2.4(1)]). Furthermore, $W(m)$ is restricted if and only if $m = 1$ (cf. [29, Theorem 4.2.4(2)]). The

Witt algebra $W(1)$ was the first non-classical simple Lie algebra discovered by Ernst Witt in the late 1930's.

Let ∂ denote the derivative defined by

$$\partial(x^{(a)}) = \begin{cases} 0 & \text{if } a = 0 \\ x^{(a-1)} & \text{if } 1 \leq a \leq p^m - 1. \end{cases}$$

Note that $W(m)$ is a free $A(m)$ -module with basis ∂ (cf. [29, Proposition 4.2.2(1)]). Set $e_i := x^{(i+1)}\partial$ for any $-1 \leq i \leq p^m - 2$. Then

$$[e_i, e_j] = \begin{cases} \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) e_{i+j} & \text{if } -1 \leq i + j \leq p^m - 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$W(m) = \bigoplus_{i=-1}^{p^m-2} \mathbb{F}e_i$$

is a \mathbb{Z} -graded Lie algebra (cf. [29, Proposition 4.2.2(2) and (3)]).

The subalgebra

$$\mathfrak{b}(m) := \bigoplus_{i=0}^{p^m-2} \mathbb{F}e_i$$

is supersolvable (i.e., every composition factor of the adjoint module of $\mathfrak{b}(m)$ is one-dimensional). Moreover, $\mathfrak{b}(m)$ is a restricted Lie algebra via the p -mapping defined by

$$e_i^{[p]} = \begin{cases} e_i & \text{if } i = 0 \\ (p-1)!e_{pi} & \text{if } i = p^t - 1 \text{ for some } 1 \leq t \leq m-1 \\ 0 & \text{otherwise} \end{cases}$$

(cf. [26, (3.2.2)]). In particular, $\mathfrak{t} := \mathbb{F}e_0$ is a one-dimensional torus of $\mathfrak{b}(m)$. Since $e_i^{[p]^2} = 0$ for every $1 \leq i \leq p^m - 2$, the p -subalgebra

$$\mathfrak{u}(m) := \bigoplus_{i=1}^{p^m-2} \mathbb{F}e_i$$

is p -unipotent so that $\mathfrak{b}(m) = \mathfrak{t} \oplus \mathfrak{u}(m)$ is strongly solvable as are the *Borel subalgebras* of Lie algebras of reductive groups.

Suppose that L is a modular Lie algebra with $C(L) = 0$. Let \mathfrak{L}_{\min} denote the p -subalgebra of $\mathfrak{gl}(L)$ generated by $\text{ad}(L)$. Then $(\mathfrak{L}_{\min}, (\cdot)^p, \text{ad})$ is a minimal p -envelope of L (cf. [29, p. 97]).

Let us apply this to the Zassenhaus algebras $W(m)$. It is clear from the \mathbb{Z} -grading of $W(m)$ that $(\text{ad } e_{-1})^{p^m} = 0$. Hence

$$\mathfrak{W}(m) = \bigoplus_{r=1}^{m-1} \mathbb{F}e_{-1}^{[p]^r} \oplus W(m)$$

is a minimal p -envelope of $W(m)$. Here e_i is identified with $\text{ad } e_i$ for every $-1 \leq i \leq p^m - 2$ while $e_{-1}^{[p]^r}$ is identified with $(\text{ad } e_{-1})^{p^r}$ for every $1 \leq r \leq m-1$. We propose to call the restricted Lie algebra $\mathfrak{W}(m)$ a *restricted Zassenhaus algebra*. It follows from [29, Theorem 4.2.4(1)] and Proposition 1.3 that $\mathfrak{W}(m)$ is semisimple and has no non-zero proper p -ideals if $p > 2$. In particular, the above p -mapping of $\mathfrak{W}(m)$ is the only one possible (cf. [29, Corollary 2.2.2(1)]). Note also that $\mathfrak{W}(m)$ is the semidirect product of a *cyclic* p -unipotent subalgebra with the ideal $W(m)$.

2 Cohomology of p -Envelopes

In this section we investigate the relationship between the ordinary cohomology of finite-dimensional Lie algebras and their finite-dimensional p -envelopes. The first lemma we need is a consequence of a long exact sequence due to Dixmier [5, Proposition 1] (cf. [24, Korollar 3.7]):

Lemma 2.1 *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of arbitrary characteristic, let M be a finite-dimensional L -module, and let I be an ideal in L of codimension 1. Then, for every non-negative integer n , there is an isomorphism*

$$H^n(L, M) \cong H^n(I, M)^L \oplus H^{n-1}(I, M)^L$$

of \mathbb{F} -vector spaces.

Proof. Because $\dim_{\mathbb{F}} L/I = 1$, there exists an element $x \in L$ such that $L = I \oplus \mathbb{F}x$. Let n be a non-negative integer. According to [5, Proposition 1], the sequence

$$H^{n-1}(I, M) \xrightarrow{u_{n-1}} H^{n-1}(I, M) \xrightarrow{s_n} H^n(L, M) \xrightarrow{r_n} H^n(I, M) \xrightarrow{u_n} H^n(I, M)$$

is exact, where u_n is induced by $\theta^n(x) : C^n(I, M) \rightarrow C^n(I, M)$ (cf. [2, Chapter I, §3, Exercise 12(a)]), s_n is induced by the connecting homomorphism, and r_n is induced by the restriction mapping from $C^n(L, M)$ to $C^n(I, M)$. Hence $\text{Ker}(s_n) = \text{Im}(u_{n-1}) = LH^{n-1}(I, M)$ and $\text{Im}(r_n) = \text{Ker}(u_n) = H^n(I, M)^L$. Consequently, one has the short exact sequence

$$(*) \quad 0 \longrightarrow H^{n-1}(I, M)/LH^{n-1}(I, M) \longrightarrow H^n(L, M) \xrightarrow{r_n} H^n(I, M)^L \longrightarrow 0.$$

It follows from the first isomorphism theorem that $H^{n-1}(I, M)/\text{Ker}(u_{n-1}) \cong \text{Im}(u_{n-1})$ and thus

$$H^{n-1}(I, M) \cong \text{Ker}(u_{n-1}) \oplus \text{Im}(u_{n-1}) \cong H^{n-1}(I, M)^L \oplus LH^{n-1}(I, M),$$

i.e., $H^{n-1}(I, M)/LH^{n-1}(I, M) \cong H^{n-1}(I, M)^L$. This in conjunction with the split short exact sequence $(*)$ yields the assertion. \square

Remark. Lemma 2.1 can also be obtained from the Hochschild-Serre spectral sequence (see [24, pp. 30–33]). Our direct proof has the advantage that the mappings in the short exact sequence $(*)$ are known explicitly.

Let $(\hat{L}, (\cdot)^p, \hat{\iota})$ denote the *universal p -envelope* of a Lie algebra L over a field of prime characteristic p (cf. [29, Definition 2, p. 92]) and let M be an L -module with corresponding representation $\rho : L \rightarrow \mathfrak{gl}(M)$. By virtue of the universal property of \hat{L} , there exists a unique restricted Lie algebra homomorphism $\hat{\rho} : \hat{L} \rightarrow \mathfrak{gl}(M)$ such that $\hat{\rho} \circ \hat{\iota} = \rho$.

Now consider an arbitrary p -envelope $(\mathfrak{L}, (\cdot)^{[p]}, \iota)$ of L . In view of [29, Proposition 2.5.6], there exists a Lie algebra homomorphism $\phi : \mathfrak{L} \rightarrow \hat{L}$ such that $\phi \circ \iota = \hat{\iota}$. Then $\hat{\rho} \circ \phi : \mathfrak{L} \rightarrow \mathfrak{gl}(M)$ is a Lie algebra homomorphism such that $(\hat{\rho} \circ \phi) \circ \iota = \rho$. In this way every L -module can be considered as an \mathfrak{L} -module (cf. [29, Theorem 5.1.1]). Note that this representation of \mathfrak{L} on M is *not* unique but depends on the choice of the Lie algebra homomorphism $\phi : \mathfrak{L} \rightarrow \hat{L}$.

It will be crucial for the proof of the main result of this section that the universal p -envelope of a modular Lie algebra L acts trivially on the cohomology of L (cf. [24, Lemma 2.10]). Slightly more generally, we will need the following result:

Lemma 2.2 *Let L be a Lie algebra over a field of prime characteristic p and let M be an L -module. If $\iota(L) \subseteq \mathfrak{H} \subseteq \mathfrak{K}$ are subalgebras of a p -envelope \mathfrak{L} of L , then $H^n(\mathfrak{H}, M)$ is a trivial \mathfrak{K} -module for every non-negative integer n .*

Proof. Let n be a non-negative integer. It is a consequence of [2, Chapter I, §3, Exercise 12(b), (3)] that $H^n(L, M)$ is a trivial L -module for every non-negative integer n . Since M is a restricted \hat{L} -module and the adjoint representation is restricted, one obtains that $H^n(L, M)$ is a restricted \hat{L} -module (cf. [2, Chapter I, §3, Exercise 12(a)] and [29, Theorem 5.2.7]). It follows that $\text{Ann}_{\hat{L}}(H^n(L, M))$ is a p -subalgebra of \hat{L} that contains $\hat{\iota}(L)$. Hence $\hat{L} = \langle \hat{\iota}(L) \rangle_p = \text{Ann}_{\hat{L}}(H^n(L, M))$, i.e., $H^n(L, M)$ is a trivial \hat{L} -module.

Now let $(\mathfrak{L}, (\cdot)^{[p]}, \iota)$ be an arbitrary p -envelope of L . Since the pullback functor from \hat{L} -modules to \mathfrak{L} -modules induced by ϕ commutes with tensor products and taking duals, the \mathfrak{L} -action on $H^n(L, M)$ factors through that of \hat{L} and therefore is also trivial.

Finally, because of $\langle \mathfrak{H} \rangle_p = \langle \iota(L) \rangle_p = \mathfrak{L}$, one can assume without loss of generality that $\mathfrak{H} = L$, hence that $\mathfrak{K} \subseteq \mathfrak{L}$ acts trivially on $H^n(\mathfrak{H}, M)$. \square

Now we derive dimension formulas for the cohomology of a finite-dimensional modular Lie algebra in terms of the cohomology of any of its finite-dimensional p -envelopes and vice versa (see [24, Satz 3.9]).

Theorem 2.3 *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of prime characteristic p , let M be a finite-dimensional L -module, and let \mathfrak{L} be a p -envelope of L with $k := \dim_{\mathbb{F}} \mathfrak{L}/L < \infty$. Then the following statements hold:*

$$(1) \quad \dim_{\mathbb{F}} H^n(\mathfrak{L}, M) = \sum_{j=0}^k \binom{k}{j} \dim_{\mathbb{F}} H^{n-j}(L, M) \text{ for every non-negative integer } n.$$

$$(2) \quad \dim_{\mathbb{F}} H^n(L, M) = \sum_{\nu=0}^n (-1)^\nu \binom{\nu+k-1}{k-1} \dim_{\mathbb{F}} H^{n-\nu}(\mathfrak{L}, M) \text{ for every non-negative integer } n.$$

Proof. (1): Choose a chain of subspaces

$$\mathfrak{L} = \mathfrak{I}_k \supset \mathfrak{I}_{k-1} \supset \cdots \supset \mathfrak{I}_1 \supset \mathfrak{I}_0 = L$$

with $\dim_{\mathbb{F}} \mathfrak{I}_l / \mathfrak{I}_{l-1} = 1$ for every $1 \leq l \leq k$. According to Proposition 1.1(2), \mathfrak{I}_l is an ideal of \mathfrak{L} for every $0 \leq l \leq k$.

Let n be a non-negative integer. It follows from Lemma 2.1 and Lemma 2.2 that

$$(**) \quad H^n(\mathfrak{I}_l, M) \cong H^n(\mathfrak{I}_{l-1}, M) \oplus H^{n-1}(\mathfrak{I}_{l-1}, M)$$

for every $0 \leq l \leq k$.

Set $a_{n,l} := \dim_{\mathbb{F}} H^n(\mathfrak{I}_l, M)$ for every non-negative integer n and every integer $0 \leq l \leq k$. Then (1) can be written as

$$a_{n,k} = \sum_{l=0}^k \binom{k}{l} a_{n-l,0} \quad \forall n \in \mathbb{N}_0$$

which we prove by induction on k . This assertion is trivial for $k = 0$, so assume it holds for some $k \geq 0$. It follows from $(**)$ and the induction hypothesis that

$$\begin{aligned} a_{n,k+1} &= a_{n,k} + a_{n-1,k} \\ &= \sum_{l=0}^k \binom{k}{l} a_{n-l,0} + \sum_{l=0}^k \binom{k}{l} a_{n-1-l,0} \\ &= a_{n,0} + \left(\sum_{l=1}^k \binom{k}{l} a_{n-l,0} \right) + \left(\sum_{l=1}^k \binom{k}{l-1} a_{n-l,0} \right) + a_{n-1-k,0} \\ &= \sum_{l=0}^{k+1} \binom{k+1}{l} a_{n-l,0}, \end{aligned}$$

whence the assertion holds also for $k + 1$.

(2): In order to prove the assertion, we first derive the identity

$$(***) \quad a_{n,l} = \sum_{\nu=0}^n (-1)^\nu a_{n-\nu,l+1} \quad \forall n \in \mathbb{N}_0, 0 \leq l \leq k-1$$

from $(**)$ and a telescoping sum argument via the observation that

$$\begin{aligned} \sum_{\nu=0}^n (-1)^\nu a_{n-\nu,l+1} &= \sum_{\nu=0}^n (-1)^\nu a_{n-\nu,l} + \sum_{\nu=0}^{n-1} (-1)^\nu a_{n-\nu-1,l} \\ &= a_{n,l} + \sum_{\nu=1}^n (-1)^\nu a_{n-\nu,l} + \sum_{\mu=1}^n (-1)^{\mu-1} a_{n-\mu,l} \\ &= a_{n,l}. \end{aligned}$$

Now we are ready to prove (2). It is a consequence of the slightly more general identity

$$a_{n,0} = \sum_{\nu=0}^n (-1)^\nu \binom{\nu+l-1}{l-1} a_{n-\nu,l} \quad \forall n \in \mathbb{N}_0, 0 \leq l \leq k$$

which can be shown by finite induction on l . For $l = 0$ the assertion is trivial, so assume it holds for some $0 \leq l \leq k-1$. It follows from the induction hypothesis and $(***)$ that

$$\begin{aligned} a_{n,0} &= \sum_{\nu=0}^n (-1)^\nu \binom{\nu+l-1}{l-1} a_{n-\nu,l} \\ &= \sum_{\nu=0}^n (-1)^\nu \binom{\nu+l-1}{l-1} \left(\sum_{\mu=0}^{n-\nu} (-1)^\mu a_{n-\nu-\mu,l+1} \right) \\ &= \sum_{\nu=0}^n \binom{\nu+l-1}{l-1} \left(\sum_{\mu=\nu}^n (-1)^\mu a_{n-\mu,l+1} \right) \\ &= \sum_{\mu=0}^n (-1)^\mu \left(\sum_{\nu=0}^{\mu} \binom{\nu+l-1}{l-1} \right) a_{n-\mu,l+1} \\ &= \sum_{\mu=0}^n (-1)^\mu \binom{\mu+l}{l} a_{n-\mu,l+1}, \end{aligned}$$

whence the assertion holds also for $l + 1$. \square

We rewrite the first part of Theorem 2.3 as an isomorphism theorem similar to a factorization theorem of Hochschild and Serre in characteristic zero (see [21, Theorem 13]).

Theorem 2.4 *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of prime characteristic p , let M be a finite-dimensional L -module, and let \mathfrak{L} be a p -envelope of L with $\dim_{\mathbb{F}} \mathfrak{L}/L < \infty$. Then, for every non-negative integer n , there is an isomorphism*

$$H^n(\mathfrak{L}, M) \cong \bigoplus_{i+j=n} \Lambda^i(\mathfrak{L}/L) \otimes_{\mathbb{F}} H^j(L, M)$$

of \mathbb{F} -vector spaces. \square

Remark. In particular, Theorem 2.4 shows that $H^n(L, M)$ is a direct summand of $H^n(\mathfrak{L}, M)$ (cf. [24, (3.15)]).

We conclude this section with some immediate consequences of Theorem 2.3 that will be useful in Section 5. The first result states that the cohomology of any finite-dimensional p -envelope of a finite-dimensional modular Lie algebra L vanishes if and only if the cohomology of L vanishes.

Corollary 2.5 *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of prime characteristic p , let M be a finite-dimensional L -module, and let \mathfrak{L} be a p -envelope of L with $\dim_{\mathbb{F}} \mathfrak{L}/L < \infty$. Then $H^j(L, M) = 0$ for every $0 \leq j \leq n$ if and only if $H^j(\mathfrak{L}, M) = 0$ for every $0 \leq j \leq n$. In particular, $H^\bullet(L, M) = 0$ if and only if $H^\bullet(\mathfrak{L}, M) = 0$. \square*

The low degree cohomology of finite-dimensional p -envelopes can be computed as follows (for the second part see also [24, Korollar 3.10]).

Corollary 2.6 *Let L be a finite-dimensional Lie algebra over a field \mathbb{F} of prime characteristic p , let M be a finite-dimensional L -module, and let \mathfrak{L} be a p -envelope of L with $k := \dim_{\mathbb{F}} \mathfrak{L}/L < \infty$. Then the following statements hold:*

- (1) $H^0(\mathfrak{L}, M) \cong H^0(L, M)$.
- (2) $\dim_{\mathbb{F}} H^1(\mathfrak{L}, M) = \dim_{\mathbb{F}} H^1(L, M) + k \dim_{\mathbb{F}} M^L$.
- (3) $\dim_{\mathbb{F}} H^2(\mathfrak{L}, M) = \dim_{\mathbb{F}} H^2(L, M) + k \dim_{\mathbb{F}} H^1(L, M) + \frac{k(k-1)}{2} \dim_{\mathbb{F}} M^L$. \square

3 Cohomological Vanishing Theorems

In this section we reformulate a cohomological vanishing theorem of Dzhumadil'daev by using the concept of a universal p -envelope. Then we derive some consequences which will become useful in the last section. By using the cohomology theory of associative algebras Farnsteiner showed that Dzhumadil'daev's vanishing theorem holds more generally for arbitrary elements of the center of the augmentation ideal of the universal enveloping algebra (see [13, Theorem 2.1 and Corollary 2.2] as well as [14] for further generalizations).

Let \mathbb{F} be a field of prime characteristic p . A polynomial of the form $f(t) = \sum_{n \geq 0} \alpha_n t^{p^n} \in \mathbb{F}[t]$ is called p -polynomial (cf. [22, Exercise V.14, p. 196]). It is clear from

$$\hat{L} = \langle L \rangle_p = \sum_{x \in L} \sum_{r \in \mathbb{N}_0} \mathbb{F} x^{p^r}$$

(cf. [29, Proposition 2.1.3(1)]) that \hat{L} is the \mathbb{F} -subspace of $U(L)$ that is generated by the evaluations of p -polynomials over \mathbb{F} in elements of L . This shows that the special elements in Dzhumadil'daev's cohomological vanishing theorem [6, Theorem 1] are exactly the elements of the center of the universal p -envelope.

Theorem 3.1 *Let L be a Lie algebra over a field of prime characteristic and let M be an L -module. If there exists an element $z \in C(\hat{L})$ such that $(z)_M$ is invertible, then $H^n(L, M) = 0$ for every non-negative integer n .*

Proof. Consider a cocycle $\psi \in Z^n(L, M)$ of degree n . According to Lemma 2.2, $\theta^n(z)(\psi)$ is a coboundary. But $z \in C(\hat{L})$ and thus $\theta^n(z)(\psi) = (z)_M \circ \psi$. Since $(z)_M$ is invertible and $B^n(L, M)$ is an \hat{L} -submodule of $C^n(L, M)$, it follows that $\psi \in B^n(L, M)$. \square

In the remainder of this section we apply Theorem 3.1 to the cohomology of simple or indecomposable modules. We begin by proving another result of Dzhumadil'daev (cf. [6, Theorem 2]).

Corollary 3.2 *Let L be a restricted Lie algebra over a field of prime characteristic and let S be a simple L -module. If $H^n(L, S) \neq 0$ for some non-negative integer, then S is a restricted L -module.*

Proof. Suppose that S is not restricted. Then there exists $x \in L$ with $(x)_S^p \neq (x^{[p]})_S$. Consider $z := x^p - x^{[p]} \in C(\hat{L})$. Since S is simple, it follows from Schur's lemma that $(z)_S$ is invertible. Hence the assertion is a consequence of Theorem 3.1. \square

Corollary 3.2 can be generalized in several directions (see [14, Theorem 6.2] or [14, Corollary 6.4]) but we will need instead an application of the following result to \mathbb{Z} -graded Lie algebras.

Corollary 3.3 *Let L be a Lie algebra over a field of prime characteristic and let M be a finite-dimensional indecomposable L -module. If there exists an ad-nilpotent element $x \in L$ such that $(x)_M$ is not nilpotent, then $H^n(L, M) = 0$ for every non-negative integer n .*

Proof. Since x is ad-nilpotent, there exists a positive integer q such that $(\text{ad}_L x)^q = 0$, i.e., $z := x^{p^e} \in C(\hat{L})$ if $p^e \geq q$. According to Fitting's lemma (cf. [22, p. 37]), $M = M_0(z) \oplus M_1(z)$ where

$$M_0(z) := \bigcup_{n \in \mathbb{N}} \text{Ker}(z)_M^n \quad \text{and} \quad M_1(z) := \bigcap_{n \in \mathbb{N}} \text{Im}(z)_M^n.$$

It follows from $z \in C(\hat{L})$ that $M_0(z)$ and $M_1(z)$ are L -submodules of M . Since $(x)_M$ is not nilpotent and M is a restricted \hat{L} -module, $(z)_M$ is also not nilpotent, i.e., $M_0(z) \neq M$. But M is indecomposable, so $(z)_M$ is invertible on $M = M_1(z)$; therefore the assertion is a consequence of Theorem 3.1. \square

Remark. Corollary 3.3 holds for arbitrary modules if one assumes that $(x)_M$ is invertible (cf. [6, Corollary 2 of Theorem 1]).

In particular, Corollary 3.3 applies to finite-dimensional \mathbb{Z} -graded Lie algebras. Let

$$L = \bigoplus_{n \in \mathbb{Z}} L_n$$

be a \mathbb{Z} -graded Lie algebra and set

$$L^+ := \bigoplus_{n>0} L_n \text{ and } L^- := \bigoplus_{n<0} L_n.$$

If L is finite-dimensional, then every element in $L^+ \cup L^-$ is ad-nilpotent so that one can apply Corollary 3.3 in order to prove the following generalization of [4, Theorem 1.1]:

Corollary 3.4 *Let L be a finite-dimensional \mathbb{Z} -graded Lie algebra over a field of prime characteristic and let M be a finite-dimensional indecomposable L -module. If there exists an element $x \in L^+ \cup L^-$ such that $(x)_M$ is not nilpotent, then $H^n(L, M) = 0$ for every non-negative integer n . \square*

Remark. Note that a simple L -module S is graded if and only if $L^+ \cup L^-$ acts nilpotently on S (see [4, Proposition 1.2(1)]).

4 Truncated Induced Modules

In this section we introduce *truncated induced modules* and *truncated coinduced modules* following [9], [10], and [16]. For applications of truncated induced modules in the structure theory of modular Lie algebras we refer the reader to [23].

Let \mathbb{F} be a field of prime characteristic p and let L be a finite-dimensional Lie algebra over \mathbb{F} . For every subalgebra K of L with finite cobasis $\{e_1, \dots, e_k\}$ there exist positive integers m_1, \dots, m_k and $v_i \in U(L)_{(p^{m_i}-1)}$ such that $z_i := e_i^{p^{m_i}} + v_i \in C(U(L))$ for every $1 \leq i \leq k$. Now consider the unital associative subalgebra $\mathcal{O}(L, K)$ of $U(L)$ generated by $K \cup \{z_1, \dots, z_k\}$.

For $\mathbf{a} \in \mathbb{N}_0^k$ we define $\mathbf{e}^\mathbf{a} := e_1^{a_1} \cdots e_k^{a_k}$ and set $\tau := (p^{m_1}-1, \dots, p^{m_k}-1)$. It follows from Jacobson's refinement of the Poincaré-Birkhoff-Witt theorem (see [22, Lemma V.4, p. 189]) that $U(L)$ is a free left and right $\mathcal{O}(L, K)$ -module with basis $\{\mathbf{e}^\mathbf{a} \mid 0 \leq \mathbf{a} \leq \tau\}$. As a consequence

$$\mathcal{O}(L, K) \cong \mathbb{F}[z_1, \dots, z_k] \otimes_{\mathbb{F}} U(K)$$

as unital associative \mathbb{F} -algebras.

Let $\sigma : K \rightarrow \mathbb{F}$ be the Lie algebra homomorphism given by $\sigma(x) := \text{tr}(\text{ad}_{L/K} x)$ for every $x \in K$. Since the mapping $\bar{\sigma}$ defined by $x \mapsto x + \sigma(x)1$ is a Lie algebra homomorphism from K into $U(K)$, there exists a unique algebra homomorphism $U(\sigma)$ from $U(K)$ into $U(K)$ that extends $\bar{\sigma}$.

Let V be a K -module. Then the action of $U(K)$ on V can be extended to $\mathcal{O}(L, K)$ by letting the polynomial algebra $\mathbb{F}[z_1, \dots, z_k]$ act via its canonical augmentation mapping. Let V_σ be the twisted module with K -action given by $x \circ v := x \cdot v + \sigma(x)v$ for every $x \in K$ and every $v \in V$. It is clear that $V_\sigma \cong V \otimes_{\mathbb{F}} \mathbb{F}_\sigma$ and that the $\mathcal{O}(L, K)$ -action on V_σ is given by $\text{id}_{\mathbb{F}[z_1, \dots, z_k]} \otimes U(\sigma)$.

The following result is a generalization of [18, Corollary 1.2] to truncated induced and coinduced modules.

Theorem 4.1 *Let L be a finite-dimensional Lie algebra over a field of prime characteristic and let K be a subalgebra of L . Then, for every K -module V , there are the following L -module isomorphisms:*

- (1) $U(L) \otimes_{\mathcal{O}(L, K)} V \cong \text{Hom}_{\mathcal{O}(L, K)}(U(L), V_{-\sigma})$.
- (2) $[U(L) \otimes_{\mathcal{O}(L, K)} V]^* \cong U(L) \otimes_{\mathcal{O}(L, K)} [V^*]_\sigma$.
- (3) $[\text{Hom}_{\mathcal{O}(L, K)}(U(L), V)]^* \cong \text{Hom}_{\mathcal{O}(L, K)}(U(L), [V^*]_{-\sigma})$.

Proof. (1) is just [16, Theorem 1.4] while (2) follows from $[U(L) \otimes_{\mathcal{O}(L,K)} V]^* \cong \text{Hom}_{\mathcal{O}(L,K)}(U(L), V^*)$ and (1). Finally, (3) is dual to (2). \square

Remark. Theorem 4.1(1) implies that $U(L) \supseteq \mathcal{O}(L, K)$ is a *free Frobenius extension of the second kind* (cf. [18, Theorem 1.1] for the restricted analogue which is a consequence of Theorem 4.1(1) and [16, Proposition 1.5]).

Let L be a finite-dimensional modular Lie algebra and let K be a subalgebra of L . Choose a basis $\{e_1, \dots, e_k, e_{k-1}, \dots, e_l\}$ of L such that $\{e_{k-1}, \dots, e_l\}$ is a basis of K . Then there exist positive integers m_1, \dots, m_l and $v_i \in U(L)_{(p^{m_i}-1)}$ such that $z_i := e_i^{p^{m_i}} + v_i \in C(U(L))$ for every $1 \leq i \leq l$. Let $I(L)$ denote the two-sided ideal of $U(L)$ generated by $\{z_1, \dots, z_l\}$, and set $\overline{U}(L) := U(L)/I(L)$. Similarly, let $I(K)$ denote the two-sided ideal of $U(K)$ generated by $\{z_{k+1}, \dots, z_l\}$, and set $\overline{U}(K) := U(K)/I(K)$.

Remark. Let L be a restricted Lie algebra and let $\chi \in L^*$. By choosing $z_i := e_i^p - e_i^{[p]} - \chi(e_i)^p \cdot 1$ for every $1 \leq i \leq l$, it can be seen that $\overline{U}(L)$ and the χ -reduced universal enveloping algebra $u(L, \chi)$ are isomorphic as unital associative \mathbb{F} -algebras (cf. [16, p. 157]).

Let V be a K -module. Then the action of $U(K)$ on V can be extended to $\overline{U}(K)$ via $z_i \cdot v := 0$ for every $k+1 \leq i \leq l$ and every $v \in V$. Since $I(K) = I(L) \cap K$, one can consider $\overline{U}(K)$ as a subalgebra of $\overline{U}(L)$ and then form induced and coinduced modules called *truncated induced modules* and *truncated coinduced modules*, respectively. In the special case of restricted Lie algebras, the truncated (co)induced modules are just (co)induced modules over reduced universal enveloping algebras. The following result (cf. [10, p. 35]) is a generalization of [16, Proposition 1.5] and can be proved in exactly the same manner.

Theorem 4.2 *Let L be a finite-dimensional Lie algebra over a field of prime characteristic and let K be a subalgebra of L . Then, for every K -module V , there are the following L -module isomorphisms:*

- (1) $U(L) \otimes_{\mathcal{O}(L,K)} V \cong \overline{U}(L) \otimes_{\overline{U}(K)} V$.
- (2) $\text{Hom}_{\mathcal{O}(L,K)}(U(L), V) \cong \text{Hom}_{\overline{U}(K)}(\overline{U}(L), V)$. \square

Remark. It is a consequence of Theorem 4.2 and Theorem 4.1(1) that $\overline{U}(L) \supseteq \overline{U}(K)$ is a *free Frobenius extension of the second kind*.

In the last section we will need Shapiro's lemma for truncated induced modules (see [16, Theorem 2.1] or [9, Theorem in §5 and Corollary 1 in §3]):

Theorem 4.3 *Let L be a finite-dimensional Lie algebra over a field of prime characteristic, let K be a subalgebra of L , and let V be a K -module. Then, for every non-negative integer n , there is an isomorphism*

$$H^n(L, U(L) \otimes_{\mathcal{O}(L,K)} V) \cong \bigoplus_{i+j=n} \Lambda^i(L/K) \otimes_{\mathbb{F}} H^j(K, V_{-\sigma})$$

of \mathbb{F} -vector spaces.

Proof. Let n be a non-negative integer. Then the following isomorphisms can be obtained from Theorem 4.1(1) in conjunction with Shapiro's lemma and the Künneth formula (cf. [30, Theorem 3.6.3]):

$$\begin{aligned}
H^n(L, U(L) \otimes_{\mathcal{O}(L, K)} V) &\cong \text{Ext}_{U(L)}^n(\mathbb{F}, U(L) \otimes_{\mathcal{O}(L, K)} V) \\
&\cong \text{Ext}_{U(L)}^n(\mathbb{F}, \text{Hom}_{\mathcal{O}(L, K)}(U(L), V_{-\sigma})) \\
&\cong \text{Ext}_{\mathcal{O}(L, K)}^n(\mathbb{F}, V_{-\sigma}) \\
&\cong \text{Ext}_{\mathbb{F}[z_1, \dots, z_k] \otimes_{\mathbb{F}} U(K)}^n(\mathbb{F} \otimes_{\mathbb{F}} V_{-\sigma}) \\
&\cong \bigoplus_{i+j=n} \text{Ext}_{\mathbb{F}[z_1, \dots, z_k]}^i(\mathbb{F}, \mathbb{F}) \otimes_{\mathbb{F}} \text{Ext}_{U(K)}^j(\mathbb{F}, V_{-\sigma}) \\
&\cong \bigoplus_{i+j=n} \Lambda^i(L/K) \otimes_{\mathbb{F}} H^j(K, V_{-\sigma}). \quad \square
\end{aligned}$$

We conclude this section by considering certain truncated induced modules over the Zassenhaus algebras. Let $\mathcal{O}(W(m), \mathfrak{b}(m))$ denote the unital associative subalgebra of $U(W(m))$ generated by $\mathfrak{b}(m)$ and $z_{-1} = e_{-1}^{p^m} \in C(\hat{W}(m))$. Moreover, for every $\lambda \in \mathbb{F}$, let $F_\lambda := \mathbb{F}1_\lambda$ be the one-dimensional $\mathfrak{b}(m)$ -module defined by $e_0 \cdot 1_\lambda := \lambda 1_\lambda$ and by $e_i \cdot 1_\lambda := 0$ for every $1 \leq i \leq p^m - 2$. Then F_λ is a unital $\mathcal{O}(W(m), \mathfrak{b}(m))$ -module via $z_{-1} \cdot 1_\lambda := 0$. In analogy with the baby Verma modules of Lie algebras of reductive groups, the truncated induced module $V(\lambda) := U(W(m)) \otimes_{\mathcal{O}(W(m), \mathfrak{b}(m))} F_\lambda$ is called a *generalized baby Verma module*.

The next result will be useful in Section 5 and is included here for the convenience of the reader. Recall that the divided power algebra $A(m)$ is a natural $W(m)$ -module via $(f\partial) \cdot g := f\partial(g)$ for all $f, g \in A(m)$.

Proposition 4.4 *Let \mathbb{F} be a field of prime characteristic p and let m be a positive integer. Then there are the following $W(m)$ -module isomorphisms:*

- (1) $W(m) \cong V(p-2)$.
- (2) $W(m)^* \cong V(1)$.
- (3) $A(m) \cong V(p-1)$.
- (4) $A(m)^* \cong V(0)$.

Proof. Set $W := W(m)$, $\mathfrak{b} := \mathfrak{b}(m)$, and $A := A(m)$.

(1): Consider the $\mathcal{O}(W, \mathfrak{b})$ -module homomorphism $\varphi : F_{p-2} \rightarrow W$ given by $1_{p-2} \mapsto e_{p^m-2}$. Then the universal property of induced modules (cf. [17, (I.3.3)]) yields a W -module homomorphism $\Phi : V(p-2) \rightarrow W$ such that $\Phi(1 \otimes 1_{p-2}) = \varphi(1_{p-2}) = e_{p^m-2}$. In particular, $\Phi \neq 0$. Since W is simple, Φ is surjective. But $\dim_{\mathbb{F}} V(p-2) = p^m = \dim_{\mathbb{F}} W$ and therefore Φ is bijective. (By employing Lemma 5.7 below, the assertion is also an immediate consequence of $\Phi \neq 0$ and Schur's lemma.)

(3): Consider the $\mathcal{O}(W, \mathfrak{b})$ -module homomorphism $\psi : F_{p-1} \rightarrow A$ given by $1_{p-1} \mapsto x^{(p^m-1)}$. Then the universal property of induced modules (cf. [17, (I.3.3)]) yields a W -module homomorphism $\Psi : V(p-1) \rightarrow A$ such that $\Psi(1 \otimes 1_{p-1}) = \psi(1_{p-1}) = x^{(p^m-1)}$. In particular, $\Psi \neq 0$. Since

$$\Psi(e_{-1}^a \otimes 1_{p-1}) = e_{-1}^a \cdot x^{(p^m-1)} = x^{(p^m-1-a)}$$

for every $0 \leq a \leq p^m - 1$, Ψ is surjective. But $\dim_{\mathbb{F}} V(p-1) = p^m = \dim_{\mathbb{F}} A$ and thus Ψ is bijective.

(2) & (4): Finally, consider the Lie algebra homomorphism $\sigma : \mathfrak{b} \rightarrow \mathbb{F}$ defined by $\sigma(x) := \text{tr}(\text{ad}_{W/\mathfrak{b}} x)$ for every $x \in \mathfrak{b}$. We have $[e_i, e_{-1}] = -e_{i-1}$ and therefore $\sigma(e_i) = -\delta_{i0}$ for every $0 \leq i \leq p^m - 2$. Hence $[F_{-\lambda}]_\sigma = F_{p-1-\lambda}$ and thus Theorem 4.1(2) yields $V(\lambda)^* \cong V(p-1-\lambda)$ for every $\lambda \in \mathbb{F}_p$. Then (2) and (4) are immediate consequences of this isomorphism in conjunction with (1) and (3), respectively. \square

Remark. It follows from the first two parts of Proposition 4.4 that $W(m)^* \cong W(m)$ if and only if $p = 3$, i.e., Zassenhaus algebras have non-degenerate invariant bilinear forms if and only if $p = 3$ (cf. [7, Theorem 1] or [29, Theorem 4.6.3]).

It is well-known that the generalized baby Verma modules of a Zassenhaus algebra $W(m)$ can be realized on the divided power algebra $A(m)$ (see [6, p. 134]). Let $\lambda \in \mathbb{F}$ and let $A_\lambda(m)$ denote the vector space $A(m)$ with $W(m)$ -action defined by $(f\partial) \cdot g := f\partial(g) + \lambda\partial(f)g$ for all $f, g \in A(m)$. As in the proof of Proposition 4.4(3), one obtains the following isomorphisms:

Proposition 4.5 *Let \mathbb{F} be a field of prime characteristic p and let m be a positive integer. Then, for every $\lambda \in \mathbb{F}$, there is a $W(m)$ -module isomorphism $V(\lambda) \cong A_{\lambda+1}(m)$. \square*

Finally, it is easy to see that the generalized baby Verma modules of Zassenhaus algebras are induced modules for restricted universal enveloping algebras of their minimal p -envelopes. Note that F_λ is a restricted $\mathfrak{b}(m)$ -module if and only if $\lambda \in \mathbb{F}_p$.

Proposition 4.6 *Let \mathbb{F} be a field of prime characteristic p and let m be a positive integer. Then, for every $\lambda \in \mathbb{F}_p$, there is a $W(m)$ -module isomorphism $V(\lambda) \cong u(\mathfrak{W}(m)) \otimes_{u(\mathfrak{b}(m))} F_\lambda$ where $z_{-1} := e_{-1}^{p^m}$. \square*

Remark. In general, there is a similar relationship between truncated (co)induced modules of modular Lie algebras $L \supseteq K$ and (co)induced modules over restricted universal enveloping algebras of certain p -envelopes $\mathfrak{L} \supseteq \mathfrak{K}$ of L and K , respectively. Namely, Farnsteiner [15, Theorem 2.3] proved that for every finite-dimensional K -module V there exist finite-dimensional p -envelopes $\mathfrak{K} \subseteq \mathfrak{L}$ of K and L , respectively, such that V is a restricted K -module and $U(L) \otimes_{\mathcal{O}(L, K)} V \cong u(\mathfrak{L}) \otimes_{u(\mathfrak{K})} V$ as L -modules.

5 Cohomology of Zassenhaus Algebras

In this section we compute the cohomology of Zassenhaus algebras with coefficients in generalized baby Verma modules and in simple modules.

The first result reduces the computation of the cohomology of the Zassenhaus algebra $W(m)$ with coefficients in a generalized baby Verma module to the \mathfrak{t} -invariants of the cohomology of the maximal nilpotent subalgebra $\mathfrak{u}(m)$ with coefficients in the one-dimensional trivial module (cf. [6, Theorem 5]).

Theorem 5.1 *Let \mathbb{F} be a field of prime characteristic p , let m be a positive integer, and let $\lambda \in \mathbb{F}$. Then, for every non-negative integer n , there is an isomorphism*

$$\begin{aligned} H^n(W(m), V(\lambda)) &\cong H^n(\mathfrak{b}(m), F_{\lambda+1}) \oplus H^{n-1}(\mathfrak{b}(m), F_{\lambda+1}) \\ &\cong [H^n(\mathfrak{u}(m), F_{\lambda+1}) \oplus H^{n-1}(\mathfrak{u}(m), F_{\lambda+1})^{\oplus 2} \oplus H^{n-2}(\mathfrak{u}(m), F_{\lambda+1})]^{\mathfrak{t}} \end{aligned}$$

of \mathbb{F} -vector spaces.

Proof. Set $W := W(m)$, $\mathfrak{b} := \mathfrak{b}(m)$, and $\mathfrak{u} := \mathfrak{u}(m)$. Let $\sigma : \mathfrak{b} \rightarrow \mathbb{F}$ be defined by $\sigma(x) := \text{tr}(\text{ad}_{W/\mathfrak{b}} x)$ for every $x \in \mathfrak{b}$. As in the proof of Proposition 4.4, we have $\sigma(e_i) = -\delta_{i0}$ for every $0 \leq i \leq p^m - 2$. Hence $[F_\lambda]_{-\sigma} = F_{\lambda+1}$ and the first isomorphism follows from Theorem 4.3. Moreover, the second isomorphism can be obtained from Lemma 2.1 and the fact that $\mathfrak{u}H^n(\mathfrak{u}, F_{\lambda+1}) = 0$. \square

Later on in this section we will need the invariant spaces of generalized baby Verma modules for the Zassenhaus algebras and their minimal p -envelopes:

Proposition 5.2 *Let \mathbb{F} be a field of prime characteristic p , let m be a positive integer, and let $\lambda \in \mathbb{F}$. Then there are isomorphisms*

$$V(\lambda)^{\mathfrak{W}(m)} \cong V(\lambda)^{W(m)} \cong \begin{cases} \mathbb{F} & \text{if } \lambda = p - 1 \\ 0 & \text{if } \lambda \neq p - 1. \end{cases}$$

of \mathbb{F} -vector spaces.

Proof. It is a consequence of Corollary 2.6(1) and Theorem 5.1 that

$$V(\lambda)^{\mathfrak{W}(m)} \cong V(\lambda)^{W(m)} \cong H^0(W(m), V(\lambda)) \cong H^0(\mathfrak{u}(m), F_{\lambda+1})^\mathfrak{t} \cong (F_{\lambda+1})^\mathfrak{t} \cong \begin{cases} \mathbb{F} & \text{if } \lambda = p - 1 \\ 0 & \text{if } \lambda \neq p - 1. \end{cases}. \quad \square$$

For $p > 3$ the following result was proved by Dzhumadil'daev (see [6, Corollary 2 of Theorem 5]) while for $p = 3$ it is contained in [24, Satz 5.17].

Theorem 5.3 *Let \mathbb{F} be a field of prime characteristic $p > 2$, let m be a positive integer, and let $\lambda \in \mathbb{F}$. Then the following statements hold:*

(1) *If $p = 3$, then*

$$\dim_{\mathbb{F}} H^1(W(m), V(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 0 \\ m - 1 & \text{if } \lambda = 1 \\ 2 & \text{if } \lambda = 2 \\ 0 & \text{if } \lambda \neq 0, 1, 2. \end{cases}$$

(2) *If $p > 3$, then*

$$\dim_{\mathbb{F}} H^1(W(m), V(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ or } \lambda = 1 \\ m - 1 & \text{if } \lambda = p - 2 \\ 2 & \text{if } \lambda = p - 1 \\ 0 & \text{if } \lambda \neq 0, 1, p - 2, p - 1. \end{cases}$$

Proof. Set $W := W(m)$ and $\mathfrak{u} := \mathfrak{u}(m)$. Since \mathfrak{u} acts trivially on $F_{\lambda+1}$, it follows from Theorem 5.1 that

$$\begin{aligned} H^1(W, V(\lambda)) &\cong H^1(\mathfrak{u}, F_{\lambda+1})^\mathfrak{t} \oplus H^0(\mathfrak{u}, F_{\lambda+1})^\mathfrak{t} \oplus H^0(\mathfrak{u}, F_{\lambda+1})^\mathfrak{t} \\ &\cong H^1(\mathfrak{u}, F_{\lambda+1})^\mathfrak{t} \oplus (F_{\lambda+1})^\mathfrak{t} \oplus (F_{\lambda+1})^\mathfrak{t}. \end{aligned}$$

For the last two summands we have that

$$(+) \quad (F_{\lambda+1})^\mathfrak{t} \cong \begin{cases} \mathbb{F} & \text{if } \lambda = p - 1 \\ 0 & \text{if } \lambda \neq p - 1. \end{cases}$$

Since \mathfrak{u} acts trivially on $F_{\lambda+1}$, we obtain for the first summand that

$$H^1(\mathfrak{u}, F_{\lambda+1})^t \cong \text{Hom}_{\mathbb{F}}(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}], F_{\lambda+1})^t,$$

where t acts via $(e_0 \cdot \eta)(\bar{x}) = e_0 \cdot \eta(\bar{x}) - \eta(\overline{[e_0, x]})$ for every $x \in \mathfrak{u}$. (Here \bar{x} denotes the coset of x in $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$.) By a straightforward computation (see [24, Lemma 5.14]) one can show that

$$\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] \cong \begin{cases} \mathbb{F}\bar{e}_1 \oplus \bigoplus_{r=1}^{m-1} \mathbb{F}\bar{e}_{3^r-1} & \text{if } p = 3 \\ \mathbb{F}\bar{e}_1 \oplus \mathbb{F}\bar{e}_2 \oplus \bigoplus_{r=1}^{m-1} \mathbb{F}\bar{e}_{p^r-1} & \text{if } p > 3. \end{cases}$$

Suppose now that $\eta \in \text{Hom}_{\mathbb{F}}(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}], F_{\lambda+1})^t$, i.e., $e_0 \cdot \eta = 0$. Then

$$0 = (e_0 \cdot \eta)(\bar{e}_i) = e_0 \cdot \eta(\bar{e}_i) - \eta(\overline{[e_0, e_i]}) = (\lambda + 1 - i)\eta(\bar{e}_i)$$

for $i \in \{1, 2, p^r - 1 \mid 1 \leq r \leq m - 1\}$. In particular, $\text{Hom}_{\mathbb{F}}(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}], F_{\lambda+1})^t = 0$ unless $\lambda \in \mathbb{F}_p$. Moreover, we obtain the following results which, in conjunction with (+), finish the proof of the theorem.

(1) If $p = 3$, then

$$\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}}(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}], F_{\lambda+1})^t = \begin{cases} 1 & \text{if } \lambda = 0 \\ m - 1 & \text{if } \lambda = 1 \\ 0 & \text{if } \lambda \neq 0, 1 \end{cases}$$

(2) If $p > 3$, then

$$\dim_{\mathbb{F}} \text{Hom}_{\mathbb{F}}(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}], F_{\lambda+1})^t = \begin{cases} 1 & \text{if } \lambda = 0 \text{ or } \lambda = 1 \\ m - 1 & \text{if } \lambda = p - 2. \\ 0 & \text{if } \lambda \neq 0, 1, p - 2 \end{cases} \quad \square$$

Remark. It is an immediate consequence of Theorem 5.3 and Proposition 4.4(1) that

$$\dim_{\mathbb{F}} H^1(W(m), W(m)) = m - 1.$$

In fact, the outer derivations of $W(m)$ are induced by $(\text{ad } e_{-1})^{p^r}$ for any $1 \leq r \leq m - 1$ (see also the remarks at the end of Section 1).

In view of the discussion before Lemma 2.2, every $W(m)$ -module is a $\mathfrak{W}(m)$ -module for the minimal p -envelope $\mathfrak{W}(m)$ of $W(m)$. In particular, $V(\lambda)$ is a $\mathfrak{W}(m)$ -module for every $\lambda \in \mathbb{F}$. The next result is a consequence of Corollary 2.6(2), Theorem 5.3, and Proposition 5.2.

Theorem 5.4 *Let \mathbb{F} be a field of prime characteristic $p > 2$, let m be a positive integer, and let $\lambda \in \mathbb{F}$. Then the following statements hold:*

(1) *If $p = 3$, then*

$$\dim_{\mathbb{F}} H^1(\mathfrak{W}(m), V(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 0 \\ m - 1 & \text{if } \lambda = 1 \\ m + 1 & \text{if } \lambda = 2 \\ 0 & \text{if } \lambda \neq 0, 1, 2. \end{cases}$$

(2) If $p > 3$, then

$$\dim_{\mathbb{F}} H^1(\mathfrak{W}(m), V(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ or } \lambda = 1 \\ m-1 & \text{if } \lambda = p-2 \\ m+1 & \text{if } \lambda = p-1 \\ 0 & \text{if } \lambda \neq 0, 1, p-2, p-1. \end{cases} \quad \square$$

Remark. According to Proposition 4.6, $V(\lambda) \cong u(\mathfrak{W}(m)) \otimes_{u(\mathfrak{b}(m))} F_\lambda$ is a restricted baby Verma module for $\mathfrak{W}(m)$ if $\lambda \in \mathbb{F}_p$. This isomorphism, in conjunction with [16, Proposition 1.5] and Theorem 4.3, can be employed to give a proof of Theorem 5.4 for $\lambda \in \mathbb{F}_p$ which does neither use Theorem 5.3 nor the results in Section 2.

Following Hochschild [20] we define the *restricted cohomology* of a restricted Lie algebra L with coefficients in a restricted L -module M by means of

$$H_*^n(L, M) := \mathrm{Ext}_{u(L)}^n(\mathbb{F}, M) \quad \forall n \in \mathbb{N}_0.$$

Observe that $V(\lambda)$ is a restricted $\mathfrak{W}(m)$ -module if and only if $\lambda \in \mathbb{F}_p$. Then Theorem 5.4 can be used to determine the dimensions of the restricted 1-cohomology of the minimal p -envelopes of the Zassenhaus algebras with coefficients in restricted baby Verma modules (cf. [17, Satz III.3.3] for $m = 1$ and $p > 3$).

Corollary 5.5 *Let \mathbb{F} be a field of prime characteristic $p > 2$, let m be a positive integer, and let $\lambda \in \mathbb{F}_p$. Then the following statements hold:*

(1) If $p = 3$, then

$$\dim_{\mathbb{F}} H_*^1(\mathfrak{W}(m), V(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 0 \\ m-1 & \text{if } \lambda = 1 \\ 0 & \text{if } \lambda = 2. \end{cases}$$

(2) If $p > 3$, then

$$\dim_{\mathbb{F}} H_*^1(\mathfrak{W}(m), V(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ or } \lambda = 1 \\ m-1 & \text{if } \lambda = p-2 \\ 0 & \text{if } \lambda \neq 0, 1, p-2. \end{cases}$$

Proof. Set $\mathfrak{W} := \mathfrak{W}(m)$, $\mathfrak{b} := \mathfrak{b}(m)$, and $\mathfrak{u} := \mathfrak{u}(m)$. According to Proposition 4.6, $V(\lambda) \cong u(\mathfrak{W}) \otimes_{u(\mathfrak{b})} F_\lambda$ is a restricted baby Verma module for \mathfrak{W} if $\lambda \in \mathbb{F}_p$. Then, from Shapiro's lemma (cf. [18, Corollary 1.4]) and the Hochschild-Serre spectral sequence for restricted cohomology in conjunction with [18, Corollary 3.6], [18, Proposition 2.7], and the definition of the p -mapping of \mathfrak{u} , one obtains that

$$\begin{aligned} H_*^1(\mathfrak{W}, V(\lambda)) &\cong H_*^1(\mathfrak{W}, u(\mathfrak{W}) \otimes_{u(\mathfrak{b})} F_\lambda) \\ &\cong H_*^1(\mathfrak{b}, F_{\lambda+1}) \\ &\cong H_*^1(\mathfrak{u}, F_{\lambda+1})^\mathfrak{t} \\ &\cong \mathrm{Hom}_{\mathfrak{t}}(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] + \langle \mathfrak{u}^{[p]} \rangle_{\mathbb{F}}, F_{\lambda+1}) \\ &\cong \mathrm{Hom}_{\mathfrak{t}}(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}], F_{\lambda+1}). \end{aligned}$$

Now the assertions follow from the dimensions formulas at the end of the proof of Theorem 5.3. \square

Remark. For $\lambda \neq p - 1$, Corollary 5.5 is an immediate consequence of Theorem 5.4 and Proposition 5.2 in conjunction with the six-term exact sequence of Hochschild [20, p. 575] or the Appendix of [19].

Let S be a simple $W(m)$ -module. According to Corollary 3.4, $H^\bullet(W(m), S) = 0$ unless e_{-1} and $\mathfrak{u}(m)$ act nilpotently on S . In the following we determine the simple modules of the Zassenhaus algebras on which e_{-1} and $\mathfrak{u}(m)$ act nilpotently (cf. [24, Lemma 5.2]).

Lemma 5.6 *Let \mathbb{F} be an algebraically closed field of prime characteristic p and let S be a simple $W(m)$ -module on which e_{-1} and $\mathfrak{u}(m)$ act nilpotently. Then there is a non-zero element $v \in S$ and there is an element $\lambda \in \mathbb{F}$ such that $e_0 \cdot v = \lambda v$ and $e_i \cdot v = 0$ for every $1 \leq i \leq p^m - 2$.*

Proof. Set $W := W(m)$ and $\mathfrak{u} := \mathfrak{u}(m)$. Because S is simple, there exists a non-zero element $w \in S$ such that $S = U(W)w$. According to [29, Theorem 3.4.3], the \mathbb{Z} -grading of W induces a \mathbb{Z} -grading on $U(W)$. Since \mathfrak{u} acts nilpotently on S , the integer $a_* := \min\{a \in \mathbb{N} \mid e_i^a \cdot S = 0 \text{ for every } 1 \leq i \leq p^m - 2\}$ exists. Then $U(W)_n \cdot w = 0$ if $n \geq a_*(p^m - 1)^2$. Namely, if $e_{-1}^{a-1} e_0^{a_0} \cdots e_{p^m-2}^{a_{p^m-2}}$ is a basis element of $U(W)_n$ with $a_i < a_*$ for every $1 \leq i \leq p^m - 2$, then

$$n = \sum_{i=-1}^{p^m-2} ia_i \leq \sum_{i=1}^{p^m-2} ia_i < a_* \sum_{i=1}^{p^m-2} i = \frac{(p^m - 2)(p^m - 1)}{2} a_* < a_*(p^m - 1)^2.$$

Now set $n_* := \max\{n \in \mathbb{Z} \mid U(W)_n \cdot w \neq 0\}$, which exists by the above, and consider $M := U(W)_{n_*} w$. Then $e_0 \cdot M \subseteq M$. Since S is finite-dimensional (cf. [29, Theorem 5.2.4]), M is also finite-dimensional. But \mathbb{F} is algebraically closed, so e_0 has an eigenvector $v \in M$ with eigenvalue $\lambda \in \mathbb{F}$. Finally, $e_i \cdot v \in U(W)_{n_*+i} \cdot w = 0$ for every $1 \leq i \leq p^m - 2$. \square

Remark. Note that a simple $W(m)$ -module S over an algebraically closed ground field is graded if and only if e_{-1} and $\mathfrak{u}(m)$ act nilpotently on S (cf. [4, Proposition 1.2(1)]).

Let S be a simple $W(m)$ -module as in Lemma 5.6. Consider $z_0 := e_0^p - e_0 \in \hat{W}(m)$. Then $[z_0, e_i] = (i^p - i)e_i = 0$ for every $-1 \leq i \leq p^m - 2$ and therefore $z_0 \in C(\hat{W}(m))$. Moreover, $z_0 \cdot v = (\lambda^p - \lambda)v$. By virtue of Theorem 3.1, $H^\bullet(W(m), S) = 0$ unless $\lambda^p = \lambda$, i.e., λ belongs to the prime field \mathbb{F}_p (cf. [24, Lemma 5.3]). Finally, consider $z_{-1} := e_{-1}^{p^m} \in C(\hat{W}(m))$ (cf. Section 1). Since S is simple, it follows from Schur's lemma and the remark after Corollary 3.3 that $H^\bullet(W(m), S) = 0$ unless $e_{-1}^{p^m} \cdot v = 0$ (cf. [24, Lemma 5.4]).

In the following let $S(\lambda)$ denote the simple $W(m)$ -module with a non-zero element $v \in S(\lambda)$ such that

$$e_{-1}^{p^m} \cdot v = 0,$$

$$e_0 \cdot v = \lambda v \text{ for some } \lambda \in \mathbb{F}_p, \text{ and}$$

$$e_i \cdot v = 0 \text{ for every } 1 \leq i \leq p^m - 2.$$

Let $r := \max\{n \in \mathbb{N}_0 \mid e_{-1}^n \cdot v \neq 0\} \leq p^m - 1$ and consider

$$M := \sum_{n=0}^r \mathbb{F} e_{-1}^n \cdot v.$$

It then follows from the Cartan-Weyl identity (cf. [29, Proposition 1.1.3(4)]) that

$$\begin{aligned}
e_i \cdot (e_{-1}^n \cdot v) &= \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} e_{-1}^j (\text{ad } e_{-1})^{n-j} (e_i) \cdot v \\
&= \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} (e_{-1}^j e_{i-n+j}) \cdot v \\
&= (-1)^{i+1} \binom{n}{n-i} (e_{-1}^{n-1-i} e_{-1})(v) \\
&\quad + (-1)^i \binom{n}{n-i} (e_{-1}^{n-i} e_0) \cdot v \\
&= (-1)^i [\lambda \binom{n}{i} - \binom{n}{i+1}] e_{-1}^{n-i} \cdot v
\end{aligned}$$

for every $0 \leq i \leq p^m - 2$ (cf. [17, Lemma III.3.1] for $m = 1$ and [24, Lemma 5.5] for the general case). Consequently, M is a non-zero submodule of $S(\lambda)$ and as the latter is simple, $M = S(\lambda)$.

Most of the generalized baby Verma modules of Zassenhaus algebras are simple (cf. [17, Lemma III.3.2a]) for $m = 1$ and [24, Lemma 5.6] for the general case).

Lemma 5.7 *If $\lambda \notin \{0, p-1\}$, then $V(\lambda)$ is a simple $W(m)$ -module.*

Proof. Set $W := W(m)$ and $\mathfrak{b} := \mathfrak{b}(m)$. Then $U(W)$ is a free right $\mathcal{O}(W, \mathfrak{b})$ -module with basis $\{e_{-1}^n \mid 0 \leq n \leq p^m - 1\}$. Hence $\{e_{-1}^n \otimes 1_\lambda \mid 0 \leq n \leq p^m - 1\}$ is a basis of $V(\lambda)$ over \mathbb{F} . As above it follows from the Cartan-Weyl identity that

$$e_i e_{-1}^n \otimes 1_\lambda = (-1)^i [\lambda \binom{n}{i} - \binom{n}{i+1}] e_{-1}^{n-i} \otimes 1_\lambda$$

for every $0 \leq i \leq p^m - 2$.

Let M be a non-zero W -submodule of $V(\lambda)$ and let $\sum_{n=s}^{p^m-1} \alpha_n e_{-1}^n \otimes 1_\lambda \in M$ with $\alpha_s \neq 0$. Then

$$\alpha_s e_{-1}^{p^m-1} \otimes 1_\lambda = \sum_{n=s}^{p^m-1} \alpha_n e_{-1}^{n+p^m-1-s} \otimes 1_\lambda = e_{-1}^{p^m-1-s} \cdot \left(\sum_{n=s}^{p^m-1} \alpha_n e_{-1}^n \otimes 1_\lambda \right) \in M.$$

But $\alpha_s \neq 0$, so $e_{-1}^{p^m-1} \otimes 1_\lambda \in M$.

Now, for every $0 \leq i \leq p^m - 2$, consider

$$\begin{aligned}
e_i e_{-1}^{p^m-1} \otimes 1_\lambda &= (-1)^i [\lambda \binom{p^m-1}{i} - \binom{p^m-1}{i+1}] e_{-1}^{p^m-1-i} \otimes 1_\lambda \\
&= (-1)^i [\lambda(-1)^i - (-1)^{i+1}] e_{-1}^{p^m-1-i} \otimes 1_\lambda \\
&= (\lambda + 1) e_{-1}^{p^m-1-i} \otimes 1_\lambda.
\end{aligned}$$

Since, by assumption $\lambda \neq p-1$, we have $\{e_{-1}^n \otimes 1_\lambda \mid 1 \leq n \leq p^m - 1\} \subseteq M$. Finally, $e_1 e_{-1} \otimes 1_\lambda = -\lambda e_{-1}^0 \otimes 1_\lambda$ and because $\lambda \neq 0$, one also obtains that $e_{-1}^0 \otimes 1_\lambda \in M$. Hence $\{e_{-1}^n \otimes 1_\lambda \mid 0 \leq n \leq p^m - 1\} \subseteq M$ and thus $M = V(\lambda)$. \square

It is a consequence of Lemma 5.7 that most of the simple modules of the Zassenhaus algebras with non-vanishing cohomology are generalized baby Verma modules (cf. [24, Satz 5.7]).

Lemma 5.8 *If $\lambda \notin \{0, p - 1\}$, then $S(\lambda) \cong V(\lambda)$ as $W(m)$ -modules.*

Proof. Set $W := W(m)$. It follows from the above computations using the Cartan-Weyl identity that the linear transformation $\Phi_\lambda : V(\lambda) \rightarrow S(\lambda)$ given by $e_{-1}^n \otimes 1_\lambda \mapsto e_{-1}^n \cdot v$ for every $0 \leq n \leq p^m - 1$ is a W -module epimorphism. Since $\Phi_\lambda \neq 0$, and by Lemma 5.7 $V(\lambda)$ is simple, it follows that $\text{Ker}(\Phi_\lambda) = 0$ whence Φ_λ is a W -module isomorphism. \square

Now it remains to consider the two cases $\lambda = 0$ and $\lambda = p - 1$. Note that, by Proposition 4.4(3) and (4), $V(p - 1) \cong A(m)$ and $V(0) \cong A(m)^*$, respectively. We begin by showing that the simple module $S(0)$ is trivial (cf. [24, Satz 5.8]).

Lemma 5.9 *$S(0)$ is isomorphic to the one-dimensional trivial $W(m)$ -module.*

Proof. Set $W := W(m)$. Suppose that $e_{-1} \cdot v \neq 0$. Because $\lambda = 0$,

$$e_i \cdot (e_{-1}^n \cdot v) = (-1)^{i+1} \binom{n}{i+1} e_{-1}^{n-i} \cdot v$$

for every $0 \leq i \leq p^m - 2$. But $\binom{n}{i+1} = 0$ unless $n - i > 0$, so $\sum_{n=1}^{p^m-1} \mathbb{F} e_{-1}^n \cdot v$ is a non-zero proper W -submodule of $S(0)$. This is a contradiction since by definition $S(0)$ is simple. Hence $S(0) = \mathbb{F}v$ with $e_i \cdot v = 0$ every $-1 \leq i \leq p^m - 2$. \square

By arguments similar to those used to establish Lemma 5.9, one can prove that $V(0)$ has a $(p^m - 1)$ -dimensional submodule. The proof of Lemma 5.7 then shows that this submodule is simple (cf. [17, Lemma III.3.2b] for $m = 1$). In particular, $V(0)$ is indecomposable.

Lemma 5.10 *The subspace $\bigoplus_{n=1}^{p^m-1} \mathbb{F}(e_{-1}^n \otimes 1_0)$ is a simple $W(m)$ -submodule of $V(0)$ and the factor module $V(0)/\bigoplus_{n=1}^{p^m-1} \mathbb{F}(e_{-1}^n \otimes 1_0)$ is isomorphic to the one-dimensional trivial $W(m)$ -module.* \square

Now we consider the case $\lambda = p - 1$. The following observation is crucial to understand the structure of $V(p - 1)$ and $S(p - 1)$ (cf. [24, Lemma 5.9]).

Lemma 5.11 *If $\lambda = p - 1$, then $e_{-1}^{p^m-1} \cdot v = 0$, i.e., $S(p - 1) = \sum_{n=0}^{p^m-2} \mathbb{F} e_{-1}^n \cdot v$.*

Proof. Set $W := W(m)$. Suppose that $e_{-1}^{p^m-1} \cdot v \neq 0$. Because $\lambda = p - 1$,

$$e_i \cdot (e_{-1}^n \cdot v) = (-1)^{i+1} \binom{n+1}{i+1} e_{-1}^{n-i} \cdot v$$

for every $0 \leq i \leq p^m - 2$. In particular, for $n = p^m - 1$ one has that

$$e_i \cdot (e_{-1}^{p^m-1} \cdot v) = (-1)^{i+1} \binom{p^m}{i+1} e_{-1}^{p^m-1-i} \cdot v = 0$$

for every $0 \leq i \leq p^m - 2$ and additionally $e_{-1} \cdot (e_{-1}^{p^m-1} \cdot v) = e_{-1}^{p^m} \cdot v = 0$. Hence $\mathbb{F}(e_{-1}^{p^m-1} \cdot v)$ is a non-zero proper W -submodule of $S(p - 1)$, a contradiction since by definition $S(p - 1)$ is simple. \square

Remark. It follows from Lemma 5.13 below that $\dim_{\mathbb{F}} S(p - 1) = p^m - 1$ and therefore

$$S(p - 1) = \bigoplus_{n=0}^{p^m-2} \mathbb{F} e_{-1}^n \cdot v.$$

The generalized baby Verma module $V(p-1)$ has a one-dimensional maximal submodule (cf. [17, Lemma III.3.2c]) for $m = 1$ and [24, Lemma 5.10] for the general case). In particular, $V(p-1)$ is indecomposable.

Lemma 5.12 $F := \mathbb{F}(e_{-1}^{p^m-1} \otimes 1_{p-1})$ is a maximal $W(m)$ -submodule of $V(p-1)$ with $W(m)F = 0$.

Proof. Set $W := W(m)$. As in the proof of Lemma 5.11, one can show that $\mathbb{F}(e_{-1}^{p^m-1} \otimes 1_{p-1})$ is a trivial W -submodule of $V(p-1)$. Suppose that M is a W -submodule of $V(p-1)$ that properly contains F and let $\sum_{n=s}^{p^m-1} \alpha_n e_{-1}^n \otimes 1_{p-1} \in M$ with $\alpha_s \neq 0$ and $0 \leq s \leq p^m - 2$. Then

$$e_{-1}^{p^m-2-s} \cdot \left(\sum_{n=s}^{p^m-1} \alpha_n e_{-1}^n \otimes 1_{p-1} \right) = \alpha_s e_{-1}^{p^m-2} \otimes 1_{p-1} + \alpha_{s+1} e_{-1}^{p^m-1} \otimes 1_{p-1}.$$

Since $F \subseteq M$ and $\alpha_s \neq 0$, it follows that $e_{-1}^{p^m-2} \otimes 1_{p-1} \in M$. But

$$e_{p^m-2} \cdot (e_{-1}^{p^m-2} \otimes 1_{p-1}) = (-1)^{p^m-1} \binom{p^m-1}{p^m-1} e_{-1}^0 \otimes 1_{p-1} = e_{-1}^0 \otimes 1_{p-1},$$

i.e., $e_{-1}^0 \otimes 1_{p-1} \in M$; therefore

$$e_{-1}^n \otimes 1_{p-1} = e_{-1}^n (e_{-1}^0 \otimes 1_{p-1}) \in M$$

for every $0 \leq n \leq p^m - 1$. Hence $M = V(p-1)$. \square

Finally, we prove that $S(p-1)$ is a factor module of the generalized baby Verma module $V(p-1)$ (cf. [17, Lemma III.3.2c]) for $m = 1$ and [24, Satz 5.11] for the general case).

Lemma 5.13 $S(p-1) \cong V(p-1)/F$ as $W(m)$ -modules.

Proof. Set $W := W(m)$. It follows from the above computations, using the Cartan-Weyl identity, that the linear transformation $\Phi_{p-1} : V(p-1) \rightarrow S(p-1)$ given by $e_{-1}^n \otimes 1_{p-1} \mapsto e_{-1}^n \cdot v$, for every $0 \leq n \leq p^m - 1$, is a non-zero W -module epimorphism. By virtue of Lemma 5.11,

$$\Phi_{p-1}(e_{-1}^{p^m-1} \otimes 1_{p-1}) = e_{-1}^{p^m-1} \cdot v = 0,$$

i.e., $F \subseteq \text{Ker}(\Phi_{p-1})$. Hence Φ_{p-1} induces a non-zero W -module epimorphism $\bar{\Phi}_{p-1}$ from $V(p-1)/F$ onto $S(p-1)$. According to Lemma 5.12, $V(p-1)/F$ is simple and thus $\bar{\Phi}_{p-1}$ is a W -module isomorphism. \square

As a consequence of Lemma 5.12 and Lemma 5.13, we obtain the short exact sequence

$$0 \longrightarrow S(0) \longrightarrow V(p-1) \longrightarrow S(p-1) \longrightarrow 0$$

of $W(m)$ -modules. Similarly, we have the short exact sequence

$$0 \longrightarrow S(p-1) \longrightarrow V(0) \longrightarrow S(0) \longrightarrow 0$$

of $W(m)$ -modules. By virtue of Theorem 4.1(2), the second short exact sequence is the dual of the first short exact sequence. But the second short exact sequence can also be obtained directly by showing that the $(p^m - 1)$ -dimensional $W(m)$ -submodule $\bigoplus_{n=1}^{p^m-1} \mathbb{F}(e_{-1}^n \otimes 1_0)$ of $V(0)$ is isomorphic to $S(p-1)$ (see the remark after Lemma 5.11).

Now we apply Theorem 5.3 in order to determine the 1-cohomology of the Zassenhaus algebras with coefficients in simple modules.

Theorem 5.14 Let \mathbb{F} be a field of prime characteristic $p > 2$, let m be a positive integer, and let $\lambda \in \mathbb{F}_p$. Then the following statements hold:

(1) If $p = 3$, then

$$\dim_{\mathbb{F}} H^1(W(m), S(\lambda)) = \begin{cases} 0 & \text{if } \lambda = 0 \\ m-1 & \text{if } \lambda = 1 \\ 2 & \text{if } \lambda = 2 \end{cases}$$

(2) If $p > 3$, then

$$\dim_{\mathbb{F}} H^1(W(m), S(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 1 \\ m-1 & \text{if } \lambda = p-2 \\ 2 & \text{if } \lambda = p-1 \\ 0 & \text{if } \lambda \neq 1, p-2, p-1 \end{cases}$$

Proof. Set $W := W(m)$ and $\mathfrak{u} := \mathfrak{u}(m)$. Then the assertions for $\lambda \neq 0, p-1$ are immediate consequences of Lemma 5.8 and Theorem 5.3. According to [29, Theorem 4.2.4(1)], W is simple and therefore perfect which yields the assertions for $\lambda = 0$. Since $H^1(W, S(0)) = 0$, we obtain the exact sequence

$$(++) \quad V(0)^W \rightarrow S(0)^W \longrightarrow H^1(W, S(p-1)) \rightarrow H^1(W, V(0)) \rightarrow 0$$

by applying the long exact cohomology sequence to the short exact sequence

$$0 \longrightarrow S(p-1) \longrightarrow V(0) \longrightarrow S(0) \longrightarrow 0.$$

Furthermore, it follows from Proposition 5.2 that $V(0)^W = 0$. Hence we obtain from (++) that

$$H^1(W, S(p-1)) \cong S(0)^W \oplus H^1(W, V(0))$$

and therefore we conclude from Lemma 5.9 and Theorem 5.3 that

$$\dim_{\mathbb{F}} H^1(W, S(p-1)) = \dim_{\mathbb{F}} S(0)^W + \dim_{\mathbb{F}} H^1(W, V(0)) = 2$$

which finishes the proof of the theorem. \square

Remark. If the ground field \mathbb{F} is algebraically closed, then Theorem 5.14 completely describes the 1-cohomology of Zassenhaus algebras with coefficients in simple modules.

In view of the discussion before Lemma 2.2, every simple $W(m)$ -module is also simple as a $\mathfrak{W}(m)$ -module. In particular, $S(\lambda)$ is a simple $\mathfrak{W}(m)$ -module for every $\lambda \in \mathbb{F}_p$. The following result is an immediate consequence of Corollary 2.6(2) and Theorem 5.14:

Theorem 5.15 Let \mathbb{F} be a field of prime characteristic $p > 2$, let m be a positive integer, and let $\lambda \in \mathbb{F}_p$. Then the following statements hold:

(1) If $p = 3$, then

$$\dim_{\mathbb{F}} H^1(\mathfrak{W}(m), S(\lambda)) = \begin{cases} m-1 & \text{if } \lambda = 0 \text{ or } \lambda = 1 \\ 2 & \text{if } \lambda = 2. \end{cases}$$

(2) If $p > 3$, then

$$\dim_{\mathbb{F}} H^1(\mathfrak{W}(m), S(\lambda)) = \begin{cases} m-1 & \text{if } \lambda = 0 \text{ or } \lambda = p-2 \\ 1 & \text{if } \lambda = 1 \\ 2 & \text{if } \lambda = p-1 \\ 0 & \text{if } \lambda \neq 0, 1, p-2, p-1. \end{cases} \quad \square$$

Remark. If S is a simple $\mathfrak{W}(m)$ -module with $H^\bullet(\mathfrak{W}(m), S) = 0$, then it follows from Corollary 3.2 that S is a restricted $\mathfrak{W}(m)$ -module. One can show that S is isomorphic to $S(\lambda)$ for some $\lambda \in \mathbb{F}_p$ by employing arguments similar to those used above in the case of $W(m)$ and by replacing the generalized baby Verma module $V(\lambda)$ by the restricted baby Verma module $u(\mathfrak{W}(m)) \otimes_{u(\mathfrak{b}(m))} F_\lambda$. As a consequence, Theorem 5.15 completely describes the 1-cohomology of $\mathfrak{W}(m)$ with coefficients in simple modules if the ground field \mathbb{F} is algebraically closed.

Now Theorem 5.15 can be used to determine the dimensions of the restricted 1-cohomology of the minimal p -envelopes of Zassenhaus algebras with coefficients in simple restricted modules (cf. [17, Satz III.3.3] for $m = 1$ and $p > 3$).

Corollary 5.16 *Let \mathbb{F} be a field of prime characteristic $p > 2$, let m be a positive integer, and let $\lambda \in \mathbb{F}_p$. Then the following statements hold:*

(1) *If $p = 3$, then*

$$\dim_{\mathbb{F}} H_*^1(\mathfrak{W}(m), S(\lambda)) = \begin{cases} 0 & \text{if } \lambda = 0 \\ m-1 & \text{if } \lambda = 1 \\ 2 & \text{if } \lambda = 2. \end{cases}$$

(2) *If $p > 3$, then*

$$\dim_{\mathbb{F}} H_*^1(\mathfrak{W}(m), S(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 1 \\ m-1 & \text{if } \lambda = p-2 \\ 2 & \text{if } \lambda = p-1 \\ 0 & \text{if } \lambda \neq 1, p-1, p-2. \end{cases}$$

Proof. The vanishing of $H_*^1(\mathfrak{W}(m), S(0))$ follows from [18, Proposition 2.7], the perfectness of $W(m)$, and the definition of the p -mapping of $\mathfrak{W}(m)$ (see Section 1). The remaining statements can be obtained from Theorem 5.15 and Proposition 5.2 in conjunction with the six-term exact sequence of Hochschild [20, p. 575] or the Appendix of [19]. \square

We conclude the paper by determining the central extensions of the Zassenhaus algebras and their minimal p -envelopes. For $p > 3$ the next result is due to Block [1, Theorem 5.1] and the corresponding non-trivial central extensions of the Zassenhaus algebras are the *modular Virasoro algebras* (cf. [8]). The case $p = 3$ is due to Dzhumadil'daev [7, Theorem 2] (cf. also [12, Theorem 3.2] and [3, Theorem 3.1 and Corollary 3.1]).

Theorem 5.17 *Let \mathbb{F} be a field of prime characteristic p and let m be a positive integer. Then the following statements hold:*

(1) *If $p = 3$, then $\dim_{\mathbb{F}} H^2(W(m), \mathbb{F}) = m-1$.*

(2) If $p > 3$, then $\dim_{\mathbb{F}} H^2(W(m), \mathbb{F}) = 1$.

Proof. Set $W := W(m)$ and $A := A(m)$.

(1): If $p = 3$, then it follows from Proposition 4.4(2) and Theorem 5.3(2) that

$$\dim_{\mathbb{F}} H^1(W, W^*) = \dim_{\mathbb{F}} H^1(W, V(1)) = m - 1.$$

According to the remark after Proposition 4.4, $W^* \cong W$ as W -modules. For any $0 \leq a \leq 3^m - 1$ let $\gamma_a : A \rightarrow \mathbb{F}$ be the linear transformation defined by

$$f = \sum_{a=0}^{3^m-1} \gamma_a(f) x^{(a)} \quad \forall f \in A.$$

Then the canonical pairing of W^* and W induces via the above isomorphism a non-degenerate invariant bilinear form on W given by $(f\partial, g\partial) := \gamma_{2^m-1}(fg)$ (cf. the proof of [29, Theorem 4.6.3]). It follows from the remark after Theorem 5.3 that the $m - 1$ linear independent outer derivations of W are induced by ∂^{3^r} for $1 \leq r \leq m - 1$. Then

$$\begin{aligned} ([\partial^{3^r}, f\partial], g\partial) + (f, [\partial^{3^r}, g\partial]) &= \gamma_{3^m-1}(\partial^{3^r}(f)g + f\partial^{3^r}(g)) \\ &= \gamma_{3^m-1}(\partial^{3^{r-1}+1}(\partial^2(f)g - \partial(f)\partial(g) + f\partial^2(g))) \\ &= 0 \end{aligned}$$

and in the light of the discussion before [7, Proposition 1], we conclude that $\dim_{\mathbb{F}} H^2(W, \mathbb{F}) = m - 1$.

(2): If $p > 3$, then it follows from Proposition 4.4(2) and Theorem 5.3(3) that

$$\dim_{\mathbb{F}} H^1(W, W^*) = \dim_{\mathbb{F}} H^1(W, V(1)) = 1$$

and the assertion can be obtained from [11, Proposition 1.3(3)] in conjunction with [29, Theorem 4.2.4(1)] and the remark after Proposition 4.4. \square

Remark. In [8, Theorem 2] Dzhumadil'daev also claims to have determined the central extensions of the Zassenhaus algebras for $p = 2$ (cf. also [3, Theorem 3.1 and Corollary 3.1]). But at least in the case $m = 1$ this result is not correct since for $p = 2$ the Witt algebra is the two-dimensional non-abelian Lie algebra which has no non-trivial central extensions.

Finally, as an immediate consequence of Corollary 2.6(3) and Theorem 5.17 we find the central extensions of the minimal p -envelopes of the Zassenhaus algebras.

Theorem 5.18 *Let \mathbb{F} be a field of prime characteristic $p > 2$ and let m be a positive integer. Then the following statements hold:*

- (1) *If $p = 3$, then $\dim_{\mathbb{F}} H^2(\mathfrak{W}(m), \mathbb{F}) = \frac{(m-1)m}{2}$.*
- (2) *If $p > 3$, then $\dim_{\mathbb{F}} H^2(\mathfrak{W}(m), \mathbb{F}) = \frac{m^2-3m+4}{2}$. \square*

Remark. From Theorem 5.18 and [7, Proposition 1] one obtains lower bounds for the dimensions of the first coadjoint cohomology $H^1(\mathfrak{W}(m), \mathfrak{W}(m)^*)$ of the minimal p -envelopes of the Zassenhaus algebras.

We leave it to the interested reader to apply Theorem 5.18 in conjunction with the six-term exact sequence of Hochschild [20, p. 575] to obtain upper bounds for the dimensions of the second restricted cohomology of $\mathfrak{W}(m)$ with coefficients in the one-dimensional trivial module. It would be interesting to know the exact dimensions of $H_*^2(\mathfrak{W}(m), \mathbb{F})$.

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